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MULTIPLE VACUA FOR NON-ABELIAN LATTICE GAUGE THEORIES ^{*}

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The various formulations of gauge theories characterized by the parameter θ are constructed for the lattice version of these theories. We do not rely on the existence of topologically stable solutions of the classical equations. These constructions are based on the existence of inequivalent representations of the canonical commutation relations.

1. Introduction

The Hilbert space spanned by the eigenstates of a Hamiltonian invariant under a discrete set of transformations, T^n , may be decomposed into a sum of subspaces, such that in each subspace the states transform with a definite phase,

$$T|\alpha; \theta\rangle = e^{2\pi i\theta} |\alpha; \theta\rangle. \quad (1.1)$$

As long as all allowed operators (observables) commute with T , the different θ sectors do not communicate with each other. On the classical level such a periodicity is associated with topologically stable solutions of the corresponding Euclidean theory. The discovery of such solutions [1] for Yang-Mills theories has focussed attention on the existence of such θ sectors in these models [2].

A different approach to the study of these theories has been to work in a space-time or space lattice [3,4], rather than in a continuum. We address ourselves to the question how to generalize the expressions for topological quantities to the lattice and thus introduce the different θ theories. It is important to maintain invariance under local gauge transformations.

Our approach will be based on a sequence of observations. In continuum situations the quantum mechanics of a particular θ sector of a theory specified by an

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action S_0 is equivalent to the $\theta = 0$ sector of another theory specified by an action $S_0 - \theta S_1$. S_1 is related to topological invariants; a property which we shall not use here. The property of interest is that S_1 is a time integral over a total time derivative. On a classical level S_0 and $S_0 - \theta S_1$ are related by canonical transformations. On a quantum level, such a transformation is achieved by different realizations of canonical commutation relations. It is this property we shall use in obtaining different θ theories on a lattice. This observation, that reference to Euclidean solutions or topological invariants is not necessary for the understanding of multiple vacua, has been made previously in the case of two-dimensional Abelian theories [5].

The difficulty in extending these ideas to lattice non-Abelian theories is that it is not immediately possible to write the generator of the aforementioned contact transformations, or the different realizations of the quantum mechanical commutation relations, in a form invariant under local gauge transformations. The generator of the contact transformations in the continuum case must first be cast into a form manifestly invariant under local gauge transformations, with terms affected only by global transformations appearing explicitly as surface integrals. A lattice analogue can then be obtained immediately.

The fact that differing physical systems may have seemingly identical Hamiltonians, as an operator is defined only when its domain is likewise specified, is illustrated by the simple example of a one-dimensional periodic potential and continuum gauge theories. No new results are obtained as this serves as an introduction to our methods. A crucial identity permitting the extension of continuum methods to the lattice is discussed in subsect. 3.2.

2. One-dimensional periodic potential

Consider a one-dimensional system specified by the Hamiltonian

$$H = \frac{1}{2}p^2 + V(\phi) , \quad (2.1)$$

with $V(\phi + 2\pi) = V(\phi)$. The usual prescription, learned in elementary quantum mechanics, is to replace p by

$$p = -i \frac{\partial}{\partial \phi} , \quad (2.2)$$

and let all operators act on the Hilbert space of functions periodic in ϕ with period 2π . The spectrum of p consists of all integers. However, there is no reason for choosing periodic functions; we could as well take our Hilbert space to consist of functions acquiring a phase under a translation of ϕ by 2π .

$$\psi(\phi + 2\pi) = e^{2mi\theta} \psi(\phi) . \quad (2.3)$$

The spectrum of p now consists of $n + \theta$, where n is an integer. This new realization can be viewed in a somewhat different manner. The only *a priori* relation we know

between p and ϕ is the commutation relation

$$[p, \phi] = -i. \quad (2.4)$$

Eq. (2.2) is not the only realization of such commutation relations, for we may make the identification

$$p = -i \frac{\partial}{\partial \phi} + \theta, \quad (2.5)$$

and operate on periodic functions. (θ could be a function of ϕ , but we will not be concerned with this more general situation.) Another way of obtaining eq. (2.5) is first to perform a contact transformation on the classical Hamiltonian,

$$\begin{aligned} p &\rightarrow P = p + \theta, \\ \phi &\rightarrow \Phi = \phi, \end{aligned} \quad (2.6)$$

and then replace p by the realization of eq. (2.2). The generating function for the above transformation is [6]

$$F_2(P, \phi) = \phi P - \theta \phi. \quad (2.7)$$

The action corresponding to the Hamiltonian acquires an additional term:

$$S_\theta = S_0 - \theta \int \dot{\phi} dt. \quad (2.8)$$

Our reason for belaboring these rather trivial points is that finding the analogues of eq. (2.7) is a sure way of finding different realizations of the basic commutation relations without upsetting any of them. Bypassing the classical level, this new realization may be obtained by the transformation

$$-i \frac{\partial}{\partial \phi} \rightarrow U^{-1}(\phi) \left[-i \frac{\partial}{\partial \phi} \right] U(\phi), \quad U(\phi) = \exp(-i\theta \phi). \quad (2.9)$$

Despite appearances, $U(\phi)$ is not an unitary transformation for it connects different Hilbert spaces.

In order to fix the theory completely, we must specify θ in addition to specifying the Hamiltonian. To determine some relations between the differing θ worlds it is useful to study the evolution kernel for the various theories corresponding to a given Hamiltonian. Let us first consider the situation where the position variable ϕ is unconstrained and allowed to vary between plus and minus infinity. For this case let

$$G(\phi', \phi; t) = \langle \phi' | e^{-iHt} | \phi \rangle. \quad (2.10)$$

It is easy to check that for the periodic case where $-\pi < \phi < \pi$ and a given θ we obtain

$$G_\theta(\phi', \phi; t) = \sum e^{2\pi i n \theta} G(\phi' + 2\pi n, \phi; t). \quad (2.11)$$

It was noted in sect. 1 that as long as we are interested in operators having the periodicity of the potential, the different θ worlds do not communicate with each other. If on the other hand, our algebra of operators includes non-periodic ones, the Hilbert space of states consists of a sum of the θ spaces. Eq. (2.11) may be inverted,

$$G(\phi', \phi; t) = \int_0^1 d\theta G_\theta(\phi', \phi; t). \quad (2.12)$$

Finally, note that the operator $U(\phi)$ of eq. (2.9) is *not* periodic; part of the difficulty in the construction of lattice gauge theories will be in the determination of such aperiodic variables.

3. SU(2) gauge theory

3.1. Continuum theory

As the easiest formulation of a lattice gauge Hamiltonian is in the $A_0 = 0$ gauge, we will, likewise, discuss the continuum theory in this gauge. With

$$\begin{aligned} A_i &= \frac{g\sigma_\alpha A_i^\alpha}{2i}, \\ E_i &= \frac{\sigma_\alpha E_i^\alpha}{2ig}, \\ B_i &= F_{jk} = \partial_j A_k - \partial_k A_j + [A_j, A_k], \\ [E_i^\alpha(\mathbf{x}), A_j^\beta(\mathbf{y})] &= -i\delta_{ij}\delta^{\alpha\beta}\delta(\mathbf{x} - \mathbf{y}), \end{aligned} \quad (3.1)$$

the Hamiltonian is

$$H = - \int d\mathbf{r} \operatorname{Tr} \left[g^2 E^2 + \frac{1}{g^2} B^2 \right]. \quad (3.2)$$

This Hamiltonian commutes with the generators of time-independent gauge transformations

$$t[\Lambda] = g \int d\mathbf{r} \operatorname{Tr} E \cdot D\Lambda. \quad (3.3)$$

Λ is a position dependent matrix and D is the covariant derivative

$$D_i \Lambda = \partial_i \Lambda + [A_i, \Lambda]. \quad (3.4)$$

Integration by parts yields

$$t[\Lambda] = -g \int d\mathbf{r} \operatorname{Tr} \Lambda D \cdot E + g \int d\mathbf{S} \cdot E \Lambda. \quad (3.5)$$

In order to recover the usual equations of motion we require $\mathbf{D} \cdot \mathbf{E} = 0$ as a condition on the states. The effect of the boundary term is open. In fact it is the specification of the behavior of states under gauge transformations on large surfaces that specifies the various θ sectors. Specifically, let $\alpha(r)$ be a smooth function of the radius vector with $\alpha(0) = 0$ and $\alpha(\infty) = 2\pi$ and in analogy with one-dimensional translations by 2π define

$$T = \exp\{t[\alpha(r)\frac{1}{2}\boldsymbol{\sigma} \cdot \hat{r}]\} , \quad (3.6)$$

with t defined in eq. (3.3). The θ Hilbert spaces consist of states defined as in eq. (1.1). In a field representation this space is made up of functionals $\psi_\theta[A_i]$ with

$$\psi_\theta[T^{-1}A_iT] = e^{2\pi i\theta} \psi_\theta[A_i] . \quad (3.7)$$

Next, we wish to find the analogue of the operator U of eq. (2.9) which connects the different θ spaces. Let

$$W[A] = \frac{1}{8\pi^2} \epsilon_{ijk} \text{Tr} \int d\mathbf{r} [A_i \partial_j A_k + \frac{2}{3} A_i A_j A_k] , \quad (3.8)$$

then

$$U[A] = \exp\{\theta W[A]\} ; \quad (3.9)$$

namely if $\psi_0[A_i]$ is periodic under the transformation of eq. (3.7) then $U[A] \psi_0[A]$ obtains the phase $2\pi\theta$ under the same transformation. Choosing a definite θ restricts the values of the operator $\pi \int dS_i E_i^a \hat{r}_a$ to $n + \theta$, with n integral. As in the one-dimensional potential problem we may insist on working in the sector of periodic states ($\theta = 0$) at the price of changing the operators for the electric fields E . With the aid of $U[A]$ of eq. (3.9) we transform

$$E \rightarrow E' = U^{-1}[A] E U[A] , \quad (3.10)$$

which gives the same results as the classical contact transformation

$$E \rightarrow E' = E - \frac{\theta}{8\pi^2} B . \quad (3.11)$$

Summarizing; a theory with the Hamiltonian of eq. (3.2) in a θ sector of Hilbert space (as defined in eq. (3.7)) is equivalent to one defined on the $\theta = 0$ sector with E in the Hamiltonian replaced by E' of eq. (3.11). Note that $\mathbf{D} \cdot \mathbf{E} = 0$ and $\mathbf{D} \cdot \mathbf{E}' = 0$ are equivalent as, by the Bianchi identities, $\mathbf{D} \cdot \mathbf{B} = 0$. The new Hamiltonian may be obtained from an action

$$S = S_0 - \frac{\theta}{4\pi^2} \epsilon_{ijk} \text{Tr} \int dt d\mathbf{r} [\dot{A}_i \partial_j A_k + \dot{A}_i A_j A_k] , \quad (3.12)$$

which up to surface terms is the same as

$$S = S_0 - \frac{\theta}{16\pi^2} \int d^4x \text{Tr} F \tilde{F} . \quad (3.13)$$

3.2. Reformulation of $W[A]$

All the previous discussion is standard and served as an introduction to the procedure for lattice theories. To follow the previous steps we will need the lattice analogue of $W[A]$. Besides requiring that it has the correct continuum limit, we also insist that it be invariant under local lattice gauge transformations. The form of $W[A]$ given in eq. (3.8) does not have an obvious lattice version satisfying the above criteria.

In this section we shall obtain an expression for $W[A]$ consisting of a volume term which is manifestly invariant under all gauge transformations and a surface term which changes appropriately under global gauge transformations.

Let us separate the coordinates of each point into its z -component and a two-dimensional vector \mathbf{r}_\perp , perpendicular to z . Define

$$V^{-1}(\mathbf{r}_\perp, z) = \text{P exp} \int_{-\infty}^z A_3(\mathbf{r}_\perp, z') dz, \quad (3.14)$$

where P denotes a path ordering. It is now a matter of integration by parts to show that

$$\begin{aligned} 8\pi^2 W[A] = & \frac{1}{2} \epsilon_{ij} \text{Tr} \int d^3 \mathbf{r}_\perp dz dz' \epsilon(z' - z) \\ & \times \{ V^{-1}(\mathbf{r}_\perp, z') F_{3i}(\mathbf{r}_\perp, z') V(\mathbf{r}_\perp, z') V^{-1}(\mathbf{r}_\perp, z) F_{3j}(\mathbf{r}_\perp, z) V(\mathbf{r}_\perp, z) \} \\ & + \text{Tr} \int dz \oint d\mathbf{r}_\perp A_3(\mathbf{r}_\perp, z) [A_i(\mathbf{r}_\perp, z) - V(\mathbf{r}_\perp, z) \partial_i V^{-1}(\mathbf{r}_\perp, z)] |_{|\mathbf{r}_\perp| \rightarrow \infty} \\ & + \epsilon_{ij} \text{Tr} \int d^2 \mathbf{r}_\perp \{ \partial_i V(\mathbf{r}_\perp, \infty) V^{-1}(\mathbf{r}_\perp, \infty) A_j(\mathbf{r}_\perp, \infty) \\ & + V^{-1}(\mathbf{r}_\perp, \infty) \partial_i V(\mathbf{r}_\perp, \infty) A_j(\mathbf{r}_\perp, -\infty) + V^{-1}(\mathbf{r}_\perp, \infty) A_i(\mathbf{r}_\perp, \infty) V(\mathbf{r}_\perp, \infty) \\ & \times A_j(\mathbf{r}_\perp, -\infty) \} - \epsilon_{ij} \text{Tr} \int d^2 \mathbf{r}_\perp dz \{ (V^{-1}(\mathbf{r}_\perp, z) \partial_i V(\mathbf{r}_\perp, z)) \\ & \times (V^{-1}(\mathbf{r}_\perp, z) \partial_j V(\mathbf{r}_\perp, z)) (V^{-1}(\mathbf{r}_\perp, z) \partial_3 V(\mathbf{r}_\perp, z)) \}. \end{aligned} \quad (3.15)$$

The first term, a volume integral is manifestly gauge invariant. The next two terms are surface integrals while the last, although at first glance appears to depend on all of space, can be brought to the form of a surface integral. This is obvious in that this term has the form of the winding number of a pure gauge field and therefore depends only on the value of its variables on the surface. More specifically, note that V is a unitary matrix and therefore may be written as

$$V(\mathbf{r}) = \exp[i\boldsymbol{\sigma} \cdot \boldsymbol{\beta}(\mathbf{r})], \quad (3.16)$$

with $\boldsymbol{\beta}(\mathbf{r})$ regular. Evaluating the last term in eq. (3.15) yields a surface integral

$$\epsilon_{ij} \text{Tr} \int d\mathbf{r} (V^{-1} \partial_i V) (V^{-1} \partial_j V) (V^{-1} \partial_3 V)$$

$$= \int dS_k \epsilon_{kij} \left(\frac{1}{2} \beta - \frac{1}{4} \sin 2\beta \right) \hat{\beta} \cdot (\partial_i \hat{\beta} \wedge \partial_j \hat{\beta}) . \quad (3.17)$$

As in the one-dimensional case we have to consider functions of “angles” rather than of periodic variables.

3.3. Lattice theory

Hamiltonian lattice gauge theories may be viewed as a collection of interacting symmetric tops defined on the links of the lattice [4]. At each lattice point r we associate a set of right-handed coordinate axes and denote the respective links by (r, \hat{i}) . On each link we define the matrix

$$u(r, \hat{i}) = \exp a A_{\hat{i}}(r) , \quad (3.18)$$

with a denoting the lattice constant and $A_{\hat{i}}(r)$ defined in eq. (3.1). Further, on each link we define two sets of angular momenta $L_{\alpha}(r, \hat{i})$ and $\bar{L}_{\alpha}(r, \hat{i})$ with the following commutation relations

$$\begin{aligned} [L_{\alpha}(r, \hat{i}), u(r', \hat{j})] &= \delta_{r,r'} \delta_{\hat{i}\hat{j}} \frac{1}{2} \sigma_{\alpha} u(r, \hat{i}) , \\ [\bar{L}_{\alpha}(r, \hat{i}), u(r', \hat{j})] &= \delta_{r,r'} \delta_{\hat{i}\hat{j}} \hat{j} u(r, \hat{i}) \frac{1}{2} \sigma_{\alpha} . \end{aligned} \quad (3.19)$$

The L 's and \bar{L} 's have the usual angular momentum commutation relations among themselves and commute with each other. The Hamiltonian of the theory is

$$\begin{aligned} H &= \sum_{r,\hat{i}} \frac{g^2}{2a} L^2(r, \hat{i}) + V , \\ V &= - \sum_{r,\hat{i} \neq \hat{j}} \frac{1}{ag^2} \text{Tr} [u(r, \hat{i}) u(r + \hat{i}, \hat{j}) u^{-1}(r + \hat{j}, \hat{i}) u^{-1}(r, \hat{j})] . \end{aligned} \quad (3.20)$$

The generator of gauge transformation which commutes with the Hamiltonian is

$$t[\Lambda(r)] = \sum_{r,\hat{i}} \Lambda(r) \cdot [L(r, \hat{i}) - \bar{L}(r, \hat{i})] . \quad (3.21)$$

In the above it should be understood that some of the link variables emanating from points on the surface of the lattice do not exist. In the interior of the lattice we again choose Gauss' law as a condition on the states

$$\sum_{\hat{i}} [L_{\alpha}(r, \hat{i}) - \bar{L}_{\alpha}(r, \hat{i})] = 0 . \quad (3.22)$$

As in the continuum theory we choose the $\Lambda(r)$'s on the surface to be rotations by 2π with the isotopic axis of rotation normal to the lattice surface. The L 's and \bar{L} 's along links connected to surface points and pointing along the rotation axes may be diagonalized and the eigenstates of H will be invariant under these rotations. This corresponds to a $\theta = 0$ sector.

In order to construct theories corresponding to other θ sectors we need the lattice analogue of $W[A]$ of eq. (3.8). We require that it be invariant under all gauge transformations restricted to interior points of the lattice. The $W[A]$ in the form of eq. (3.8) does not inspire an immediate lattice version. However, the form of eq. (3.15) does. We define

$$F_{ij}^{\alpha} = \sigma^{\alpha \frac{1}{2}} \text{Tr} [\sigma^{\alpha} u(r, \hat{i}) u(r, \hat{i}, \hat{j}) u^{-1}(r + \hat{j}, \hat{i}) u^{-1}(r, \hat{j})] ,$$

$$V_L(r, z) = \prod_{z' < z} u(r_L, z'; \hat{k}) = \exp[i \boldsymbol{\sigma} \cdot \boldsymbol{\beta}(r)] . \quad (3.23)$$

The transcription of the volume terms in eq. (3.15) is immediate and manifestly gauge invariant. As the surface terms are not gauge invariant anyway, any lattice version with a correct continuum limit will do. Define $B_{\alpha}(r, \hat{i})$ and $\bar{B}_{\alpha}(r, \hat{i})$ as the link magnetic fields:

$$[L_{\alpha}(r, \hat{i}), W] = -\frac{i}{4\pi^2} B_{\alpha}(r, \hat{i}) ,$$

$$[\bar{L}_{\alpha}(r, \hat{i}), W] = -\frac{1}{4\pi^2} \bar{B}_{\alpha}(r, \hat{i}) . \quad (3.24)$$

The explicit forms of the B 's and \bar{B} 's are extremely complicated and related in a very non-local way to the link variables $u(r, \hat{i})$. They do, however, have two nice properties. They transform under gauge rotations as their corresponding L 's or \bar{L} 's and they satisfy a lattice version of the Bianchi identities

$$\sum_{\hat{i}} [B_{\alpha}(r, \hat{i}) - \bar{B}_{\alpha}(r, \hat{i})] = 0 . \quad (3.25)$$

Thus, the θ theory is constructed by replacing $L' = L + (\theta/4\pi^2)B$ for L in the Hamiltonian, eq. (3.20), while maintaining that the states are invariant under all transformations generated by the $t[\Lambda]$ of eq. (3.21) with the original angular momentum variables and *not* the transformed ones. As both the old and transformed angular momenta have the same spectra, namely $j(j+1)$, the question as to which θ sector we are in is determined by which operators we use to define the gauge transformations. Due to the identities of eq. (3.25) this is irrelevant for the interior points of the lattice but makes all the difference on the surface.

Finally, it might be tempting to introduce the θ theories on a lattice by adding a lattice version of $\text{Tr } F\tilde{F}$, using only local forms of F , as in eq. (3.21), to a lattice Lagrangian. This would have the correct continuum form, but for any finite lattice constant this term would not be a time derivative, or even a lattice finite time difference. We have no answer to the question whether the continuum limits of the two quantum theories are the same.

As a last point we mention that, as in eq. (2.12), we may integrate over θ in the evolution kernel and obtain a theory with no periodicity at all.

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